Analysis of dominant uncertainty in the calibration of optical tachometers.

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***Abstract* — The mathematical model of uncertainty estimation in the calibration of optical tachometers is presented. Emphasis is placed on the analysis of the dominant uncertainty component, for which two mathematical models are used and compared to each other. The first model uses a mathematical expression that constitutes a discriminant and the second model is based on Monte Carlo methods. Results are displayed and analyzed for five calibration points.**

***Index Terms* — Calibration, frequency measurement, measurement uncertainty, metrology, Monte Carlo methods, probability distribution, tachometers.**

I. Introduction

Dominance analysis is an important criteria in the calibration of optical tachometers in a lot of cases, because of the instrument´s metrological features, dominance happens by the resolution component. For this analysis, two mathematical models (described in [1] and [2]) are considered and compared with each other.

This article is complementary to an article generated in 2016 [3]. New observations are made and the use of Monte Carlo methods are included among other metrological analyzes in the magnitude of frequency.

Note: This is the extended paper of the lecture presented at CPEM 2020 (see [5])

II. Data acquisition and results

Data were acquired by experimental set-up of optical tachometers calibration reported in [3]. Results were obtained for five calibration points as shown in table 1.

Note 1: A square type frequency signal is used.

III. Mathematical model

Three components of uncertainty are considered, namely: repeatability, resolution and standard. The second component is rectangular type while the other two are normal type.

Uncertainty is estimated using the GUM method [4]; the structure of the model is the same presented in [3], however, on this article the objective equation is given by the error instead of fractional frequency deviation. This is due to a few (around 10) data are taken in the tachometers calibration. The data processing is not related to a series of time.

The general mathematical model is

$$E = f_{IBC} - f_P. \qquad (1)$$

Where, f_{IBC} is the frequency measured by the instrument under calibration and f_P is the frequency generated by the standard instrument.

A. Discriminant model

With the uncertainty components identified by mathematical model (1), it is important to point out that an option to assign the freedom degrees to the components with a type B evaluation is through equation G.3 of [4]. It has

$$\nu = \frac{1}{2}\left(\frac{\Delta u}{u}\right)^{-2}. \qquad (2)$$

Note 2: From the perspective of this article it is preferred to work with (2) instead of taking infinite freedom degrees for a rectangular type B distribution.

Having the freedom degrees associated with each uncertainty component, the Welch-Satterthwaite equation is used to obtain the effective freedom degrees. So it can get the coverage factor k and finally the expanded uncertainty [3].

Once the combined uncertainty is obtained, the dominance analysis must be done, for which, this first model poses a criteria given by

$$\frac{\sqrt{\sum_{i \neq maxi} c_i^2 u_i^2}}{\{|c_i| u_i\}_{maxi}} < 0.3\ . \qquad (3)$$

Where the subindex i is the i-th sensitivity coefficient and "maxi" refers to the uncertainty component with the greatest contribution. If (3) is fulfilled, that component is dominant. This criteria is taken from [1] (see section S9.14 of supplement 2). Note: It is good to clarify that (3) is rewritten with respect to its version in [1] and that the background of the expression is a Taylor series expansion.

Note that ultimately (3) is a discriminant. For this reason in this article, this model will be called “D model”. Applying these concepts to the case study in this article, when the value of the component by resolution is the greatest, it has

$$\frac{\sqrt{u_{REPE}^2 + u_P^2}}{u_{RESO}} < 0.3\ . \qquad (4)$$

Where, u_{REPE} is the component by repeatability, u_P is the component by the standard and u_{RESO} is the component by resolution. Note: The details of each of these three components can be consulted in [3]. For now, it is important to bear in mind that u_{RESO} has a rectangular (homogeneous) probability distribution associated with it.

Note: Annex 1 contains the data table used (real calibrations) in this model D.

B. MC Model

This model consists in obtaining the uncertainty through Monte Carlo methods. The NIST Uncertainty Machine application was used directly [2]. In this case, what is obtained is more detailed information on the probability distribution associated with the combined uncertainty. In this article, this mathematical model is called "MC Model".

It´s necessary understanding better the behavior of the MC model. The following numerical trial was performed: Data from a real calibration were taken and gradually modified to obtain different results in their standard deviation S (where S is defined according to [4]). Although the data has been intentionally modificated, these values would perfectly be obtained in a real calibration (see section IV. Results).

Note: In Annex 2, the table of data used in this numerical trial is placed; there is also a comment about it.

C. Note on the Central Limit Theorem

From the perspective of this work, the Central Limit Theorem plays a very important role. This theorem is referenced in Annex G of [4]. Certainly, it can often assume that the conditions of the Central Limit Theorem (CLT) are satisfied. In other words, it assumes that the coverage factor k is equal to 2 or close (it is obtained from the approximation t-Student). However, it is rare to do a verification of this; It may happen that there is a dominant component of uncertainty with other distribution types than the normal one and it imposes its distribution on the output distribution (it means, on the combined uncertainty). Summarizing what was stated in [4], we have the following:

Given:

$$Y = f(X_1, X_2, \ldots, X_n). \qquad (5)$$

Es decir, Y es una función que depende de variables de entrada X_i. The distribution of Y is approximately normal if the following conditions are met:

- The input variables are independent of each other.
- None of the input variables with a distribution other than Normal is dominant.

The following images illustrate what happens.

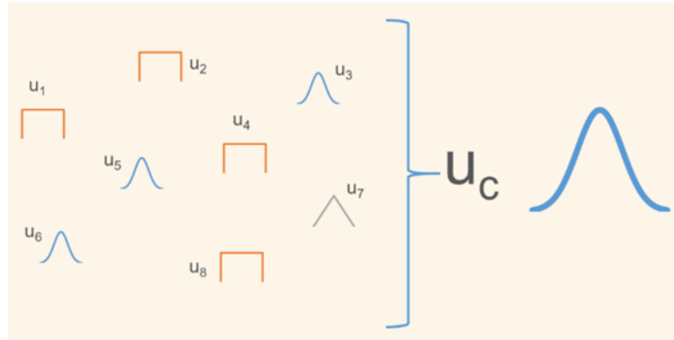


Image 1. Case in which the conditions of the CLT are satisfied. Note: uc refers to the combined uncertainty.

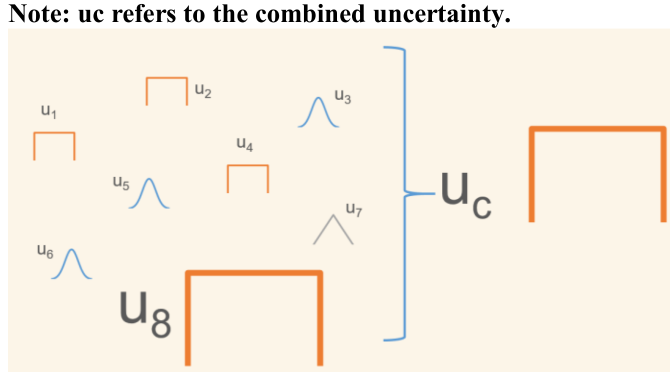


Image 2. Case of dominance of a rectangular component.

The case illustrated in image 2 is the one that interests to identify and study in this work.

Note: Annex 3 schematically explains the steps that are followed (in model D) when each of the two scenarios presented in the two previous images are given.

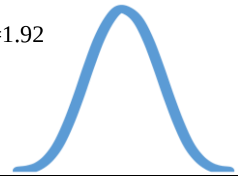
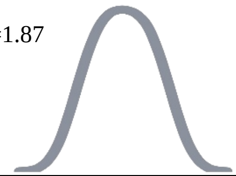
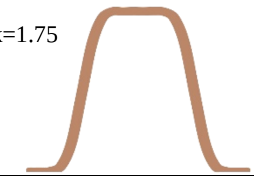
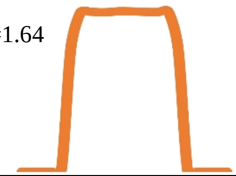
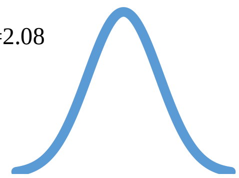
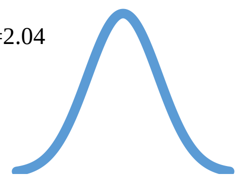
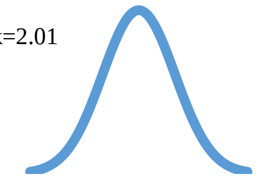
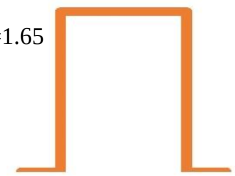
IV. Results

Table 1 shows the numerical results obtained through the D Model and the MC Model. In table, u_c and U are the combined uncertainty and the expanded uncertainty respectively, as defined in (10) and (18) of [4]. The coverage probability is 95.45%. Table 2 shows the shape of the probability distribution (and its respective coverage factor k) obtained with each model given by the numerical trial.

Table 1. Results comparison (real calibrations). Units: RPM (Coverage probability = 95.45%)

F (RPM)	D Model			MC Model		
	u_c	k	U	u_c	k	U
20	0.0327	2.02	0.0660	0.0327	1.82	0.0596
60	0.0289	1.65	0.0476	0.0289	1.65	0.0476
300	0.0470	2.12	0.0997	0.0471	1.94	0.0913
15000	0.493	2.13	1.05	0.494	1.94	0.960
99000	5.354	2.32	12.4	5.360	1.96	10.5

Table 2. Numerical trial. Probability distribution shape and coverage factor in terms of the standard deviation S. (Coverage probability = 95.45%)

Model	$S_1 = 0.95$	$S_2 = 0.63$	$S_3 = 0.32$	$S_4 = 0.00$
MC	k=1.92 	k=1.87 	k=1.75 	k=1.64 
D	k=2.08 	k=2.04 	k=2.01 	k=1.65 

V. Analysis of Results

Both models clearly show that appear the dominance of resolution component (point of 60 RPM in Table 1), which is rectangular.

In cases of non-dominance (situation in which the conditions of the central limit theorem holds) important differences are observed. It is clear that the D Model presents an over-estimation of uncertainty. This is to be expected given the background about (2).

The MC Model allows to observe a gradualness in the behavior of the probability distribution while the D Model establishes a specific limit on which it is not possible to differentiate, for example, cases of rectangular distribution strong or weakly dominant.

It should be noted that the analysis presented is not restricted to the calibration of tachometers. It has a very general applicability and can be presented in another types of calibrations.

VI. Conclusion

In the calibration of optical tachometers the mathematical concept of dominance should be considered for the estimation of uncertainty instead of applying a generic method of statistical analysis.

References

[1] EA-4/02 M:2013 Expression of the Uncertainty of Measurements in Calibration (including supplements 1 and 2 to EA-4/02)

[2] T. Lafarge and A. Possolo (2015) "The NIST Uncertainty Machine", NCSLI Measure Journal of Measurement Science, volume 10, number 3 (September), pages 20-27.

[3] L. C. Hernández and N. Bahamón, "Uncertainty Estimation in Optical Tachometers Calibration" *Momento. n. 52, pp 68-82. January 2016.*

[4] JCGM 100:2008 Evaluation of measurement data – Guide to the expression of uncertainty in measurement, BIPM, IEC, IFCC, ISO, IUPAC, OIML, (GUM 1995 with minor corrections).

[5] Conference on Precision Electromagnetic Measurements CPEM 2020. Analisys of dominant uncertainty in the calibration of optical tachometers. Nelson Bahamón, Liz Catherine Hernández Forero, Claudia Fernanda Rodríguez, Alexander Gutierrez Guevara. Instituto Nacional de Metrología de Colombia.

ANNEX 1. Real calibrations data.

The following table shows the uncertainty budget for the calculations obtained with real calibrations in model D. No explanations are included because these can be consulted in [3]

Table 3. Real calibrations data (Model D).

Data	19 RPM	60 RPM	300 RPM	15000 RPM	99000 RPM
1	19.9	60.0	300.1	15001	99011
2	19.9	60.0	300.2	15001	99020
3	20.0	60.0	299.9	14998	99001
4	19.9	60.0	299.9	15001	98962
5	19.9	60.0	300.0	15001	99011
6	19.9	60.0	299.9	15001	98991
7	20.0	60.0	300.0	15001	98991
8	19.9	60.0	300.1	14998	99011
9	20.0	60.0	300.2	15001	99011
10	19.9	60.0	300.1	15001	99011
Ave	19.9	60.0	300.0	15000	99002
Error	-0.1	0.0	0.0	0	2
S	0.0483	0.00	0.12	1.26	16.90
u_{REPE}	0.0153	0.00	0.04	0.4	5.3
u_{RESO}	0.0289	0.03	0.03	0.3	0.3
u_P	4.60E-07	1.38E-06	7E-06	3E-04	2E-03
uc	0.0327	0.0289	0.05	0.5	5.4
Discri	0.529	0.00005	0.78	0.72	0.05
v urepe	9	9	9	9	9
v ureso	200	200	200	200	200
v up	200	200	200	200	200
v uc	119	200	23	21	9
k	2.02	1.65	2.12	2.13	2.32
U	0.07	0.05	0.1	2	13

ANNEX 2. Numerical trial data.

The following is the data table used in the numerical trial mentioned in section III.B and whose graphic results are illustrated in table 2.

Table 4. Numerical trial data (units: RPM).

Data	Trial 1	Trial 2	Trial 3	Trial 4
1	99002	99002	99002	99002
2	99002	99002	99002	99002
3	99002	99002	99002	99002
4	99002	99002	99002	99002
5	99002	99002	99002	99002
6	99002	99002	99002	99002
7	99002	99002	99002	99002
8	99002	99002	99002	99002
9	99002	99002	99002	99002
10	99005	99004	99003	99002
Average	99002.30	99002.20	99002.10	99002.00
S	0.95	0.63	0.32	0.00

It can be noted that actually, the transition from normal to rectangular output distribution is subtle. In order to visualize the gradualness shown in table 2, what was done was to slightly vary only one of the data (record 10).

ANNEX 3. Detail on model D.

Due to (4) there are two scenarios. When inequality is satisfied and when not. The following image illustrates this:

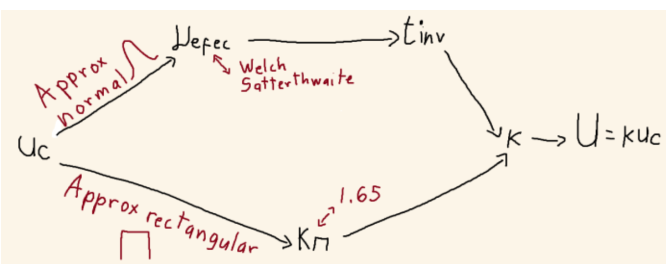


Image 3. Scheme of steps to follow according to the output distribution.

If (4) is not satisfied, it means that the conditions of the Central Limit Theorem are satisfied. In other words the output distribution is approximately normal. In this case, the calculation of the effective degrees of freedom is made using the Welch Satterthwaite equation. With this result and taking into account the chosen coverage probability, the inverse t distribution is calculated to obtain the coverage factor k and finally the expanded uncertainty. This path is the best known and most common as the dominance analysis is not generally done.

If (4) is satisfied, it means that the conditions of the Central Limit Theorem are not satisfied. We have that the component of uncertainty by resolution is dominant and therefore the output distribution is rectangular (homogeneous). This case is much less known or taken into account. Here the coverage factor k is immediate because it is associated with a rectangular distribution (see Annex G of [4].) and its value is 1.65.